Warsaw University of Technology
Faculty of Power and Aeronautical Engineering
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Institute of Aeronautics and Applied Mechanics

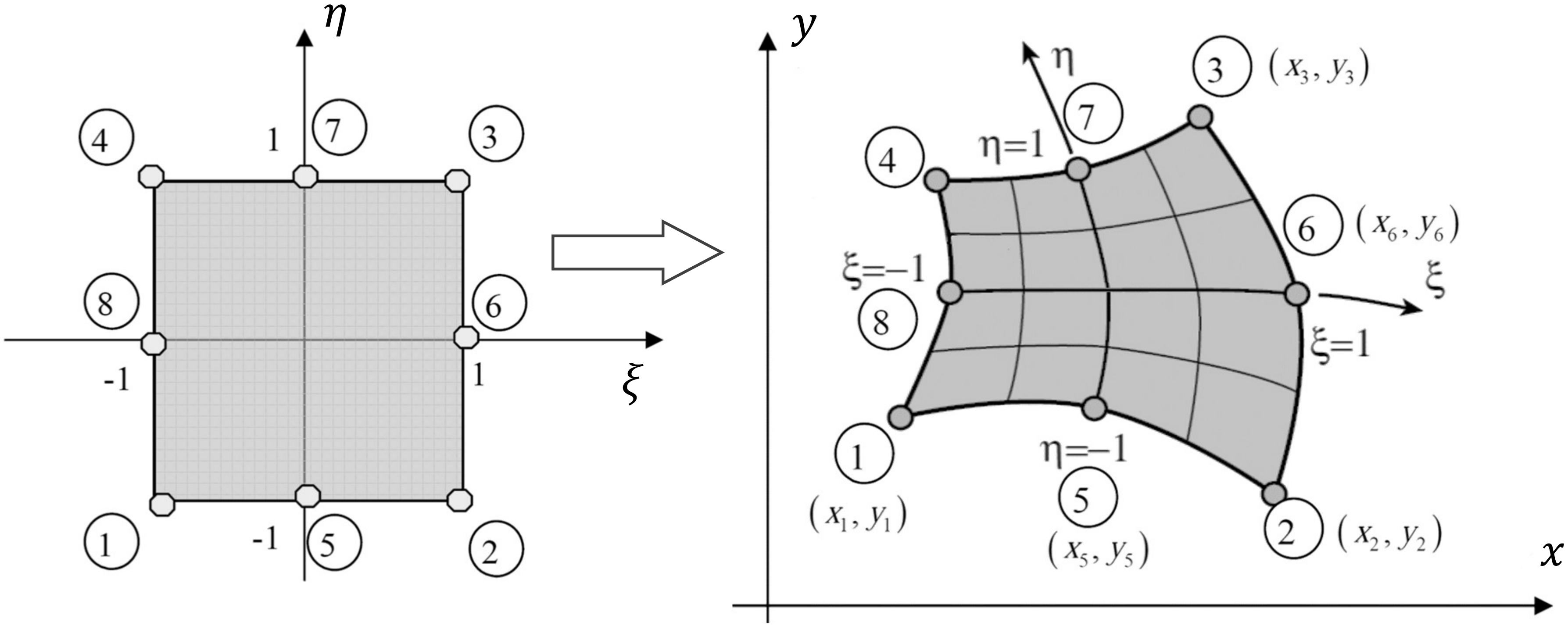
Finite element method (FEM)

8-node quadrilateral element

03.2021

8-node 2D quadrilateral element (accuracy, irregular shapes)

natural coordinate system | cartesian coordinate system



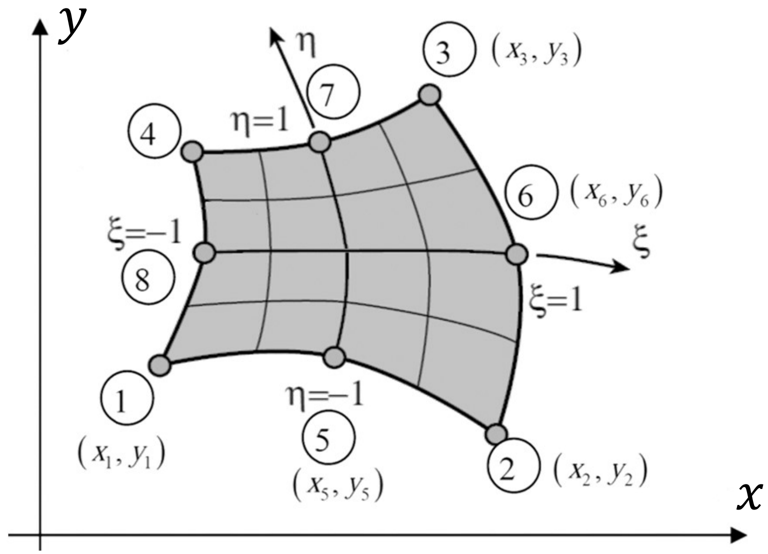
geometry mapping: $(\xi, \eta) \rightarrow (x, y)$

$(-1,-1) \rightarrow (x_1, y_1)$ $(1,-1) \rightarrow (x_2, y_2)$ $(1,1) \rightarrow (x_3, y_3)$ $(-1,1) \rightarrow (x_4, y_4)$

$(0,-1) \rightarrow (x_5, y_5)$ $(1,0) \rightarrow (x_6, y_6)$ $(0,1) \rightarrow (x_7, y_7)$ $(-1,0) \rightarrow (x_8, y_8)$

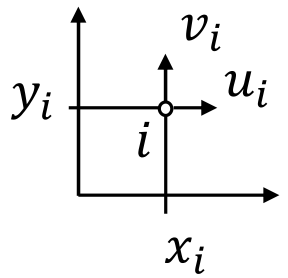
Isoparametric mapping

vectors of nodal coordinates



$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_8 \end{Bmatrix} \quad ; \quad \underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_8 \end{Bmatrix}$$

local vector of nodal parameters:



$n = 8$; $n_p = 2$ →
$n_e = n \cdot n_p = 16$

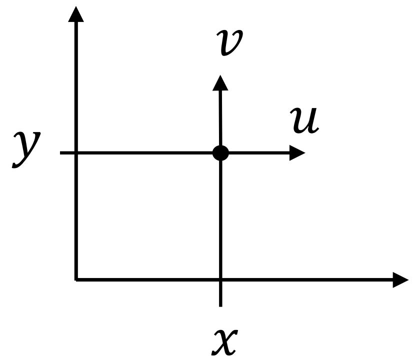
$$\underset{16\times 1}{\{q\}_e} = \begin{Bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ q_{16} \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ . \\ . \\ . \\ u_8 \\ v_8 \end{Bmatrix}_e$$

Isoparametric mapping

matrix of shape functions:

$$\underset{2\times 16}{[N(\xi,\eta)]} = \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & \dots & N_8(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & \dots & 0 & N_8(\xi,\eta) \end{bmatrix}$$

vector of shape functions:



$$\underset{1\times 8}{\lfloor N(\xi,\eta) \rfloor} = \lfloor N_1(\xi,\eta), N_2(\xi,\eta), \dots, N_8\,(\xi,\eta) \rfloor$$

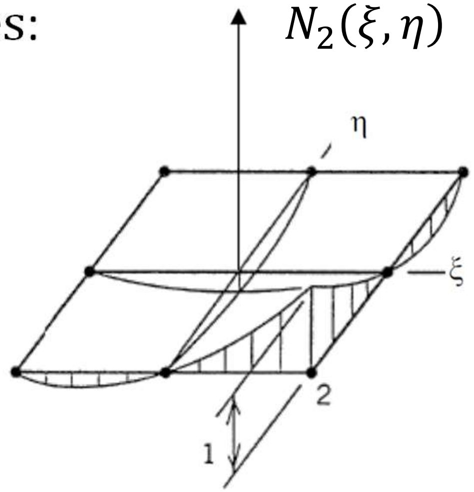
position and displacement of any point:

$$x = \underset{1\times 8}{\lfloor N(\xi,\eta) \rfloor} \underset{8\times 1}{\{x_i\}_e} \quad ; \quad y = \underset{1\times 8}{\lfloor N(\xi,\eta) \rfloor} \underset{8\times 1}{\{y_i\}_e}$$

$$\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{2\times 16}{[N(\xi,\eta)]} \underset{16\times 1}{\{q\}_e}$$

Shape functions of the 8-node quadrilateral finite elment

corner nodes:



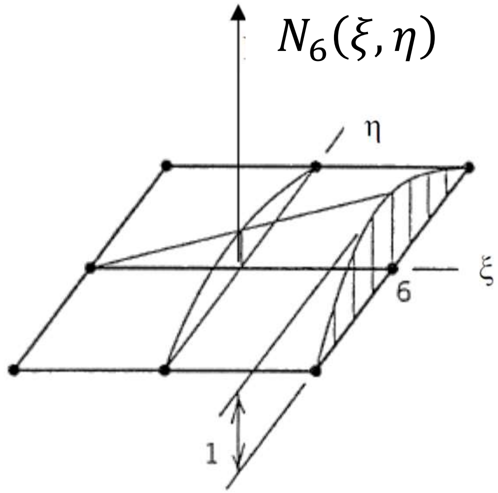
$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

midside nodes:



$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Transformation between natural and cartesian coordinates

partial derivatives of any function of coordinates (x, y) with respect to (ξ, η):

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Longrightarrow \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

$\uparrow$ *Jacobian matrix*

partial derivatives of any function of coordinates (ξ, η) with respect to (x, y):

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\Longrightarrow \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$\uparrow$ *inverse Jacobian matrix*

Transformation between natural and cartesian coordinates

differential operators:

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \quad ; \quad \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}$$

$Jacobian\ matrix$ $inverse\ Jacobian\ matrix$

differential operators:

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} \quad ; \quad \underset{2\times 2}{[J]} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[I]} \cdot \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}$$

$unit\ matrix$

Transformation between natural and cartesian coordinates

inverse Jakobian matrix:

$$\underset{2\times 2}{[J]^{-1}} = \frac{1}{det[J]}\left(\underset{2\times 2}{[J]^{C}}\right)^{T} = \frac{1}{det[J]}\left(\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}\right)^{T} = \frac{1}{det[J]}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{det[J]}\frac{\partial y}{\partial \eta} & -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} \\ -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} & \frac{1}{det[J]}\frac{\partial x}{\partial \xi} \end{bmatrix}$$

↑ $cofactors\ matrix$

$$\underset{2\times 2}{[J]^{-1}} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \rightarrow$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial \lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{y_i\}_e}$$

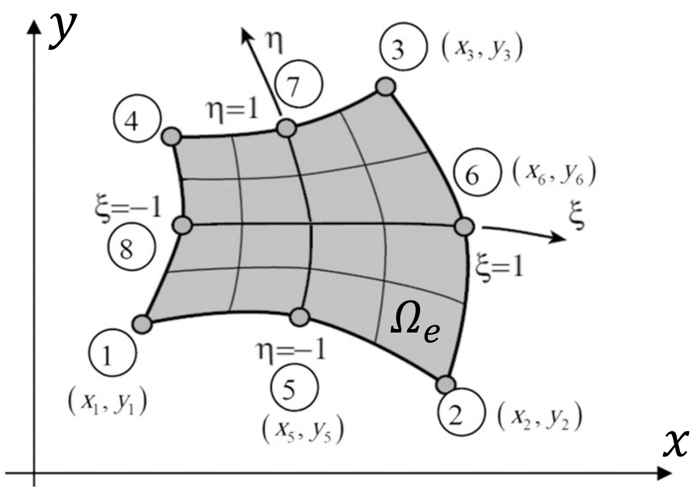
$$\frac{\partial \eta}{\partial x} = -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial \lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{y_i\}_e}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial \lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{x_i\}_e}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{det[J]}\frac{\partial x}{\partial \xi} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial \lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{x_i\}_e}$$

Transformation between natural and cartesian coordinates

determinant of the Jakobian matrix:



$$det[J] = det\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} =$$

$$= \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} - \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} =$$

$$= \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{8\times 1}{\{x_i\}_e}\right)$$

$(known\ at\ any\ point\ of\ the\ domain\ \Omega_e)$

Gradient matrix calculation

differential oparators in the coordinate system (x, y):

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]^{-1}} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{1}{det[J]}\frac{\partial y}{\partial \eta} & -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} \\ -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} & \frac{1}{det[J]}\frac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \rightarrow$$

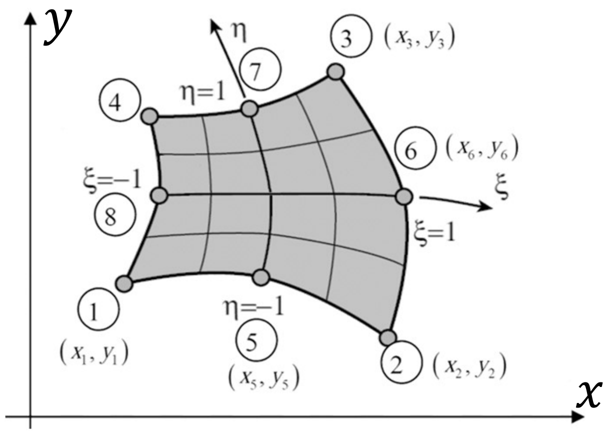
$$\frac{\partial}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \quad ; \quad \frac{\partial}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}$$

gradient matrix for plane stress or plane strain conditions:

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) & 0 \\ 0 & \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) \\ \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) & \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) \end{bmatrix} = \underset{3\times 2}{[R(\xi, \eta)]}$$

Gradient matrix for plane stress and plane strain conditions

$$\underset{3\times 2}{[R(\xi,\eta)]} = \begin{bmatrix} \dfrac{\partial}{\partial x}(\xi,\eta) & 0 \\ 0 & \dfrac{\partial}{\partial y}(\xi,\eta) \\ \dfrac{\partial}{\partial y}(\xi,\eta) & \dfrac{\partial}{\partial x}(\xi,\eta) \end{bmatrix}$$



$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_8 \end{Bmatrix} \qquad \underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_8 \end{Bmatrix}$$

$$\frac{\partial}{\partial x}(\xi,\eta) = \frac{\left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\overset{8\times 1}{\{y_i\}_e}\right)\dfrac{\partial}{\partial\xi} - \left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\overset{8\times 1}{\{y_i\}_e}\right)\dfrac{\partial}{\partial\eta}}{\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{x_i\}_e}\right)}$$

$$\frac{\partial}{\partial y}(\xi,\eta) = \frac{\left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\overset{8\times 1}{\{x_i\}_e}\right)\dfrac{\partial}{\partial\eta} - \left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\overset{8\times 1}{\{x_i\}_e}\right)\dfrac{\partial}{\partial\xi}}{\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{x_i\}_e}\right)}$$

Strain energy in the 8-node QUAD element

strain vector for plane stress or plane strain conditions:

$$\underset{3\times1}{\{\varepsilon\}} = \underset{3\times2}{[R(\xi,\eta)]}\underset{2\times1}{\{u\}} = \underset{3\times2}{[R(\xi,\eta)]}\underset{2\times16}{[N(\xi,\eta)]}\underset{16\times1}{\{q\}_e} = \underset{3\times16}{[B(\xi,\eta)]}\underset{16\times1}{\{q\}_e}$$

elastic strain energy in a finite element:

$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times3}{\lfloor\varepsilon\rfloor}\underset{3\times1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2}\underset{1\times16}{\lfloor q\rfloor_e}\left(t_e \int_{A_e} \underset{16\times3}{[B(\xi,\eta)]^T}\underset{3\times3}{[D]}\underset{3\times16}{[B(\xi,\eta)]}\, dxdy\right)\underset{16\times1}{\{q\}_e} =$$

$$\underset{3\times1}{\{\sigma\}} = \underset{3\times3}{[D]}\ \underset{3\times1}{\{\varepsilon\}}$$

$$= \frac{1}{2}\lfloor q\rfloor_e [k]_e \{q\}_e$$

$$\underset{1\times3}{\lfloor\varepsilon\rfloor} = \underset{1\times16}{\lfloor q\rfloor_e}\underset{16\times3}{[B(\xi,\eta)]^T} \qquad \underset{3\times1}{\{\varepsilon\}} = \underset{3\times16}{[B(\xi,\eta)]}\underset{16\times1}{\{q\}_e}$$

$$\left(\int_{A_e} f(x,y)\, dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta) det[J]\, d\xi d\eta\right)$$

local stiffness matrix:

$$\underset{16\times16}{[k]_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \underset{16\times3}{[B(\xi,\eta)]^T}\underset{3\times3}{[D]}\underset{3\times16}{[B(\xi,\eta)]} det[J]\, d\xi d\eta \quad (calculated\ numerically)$$

Potential energy of loading and equivalent load vector

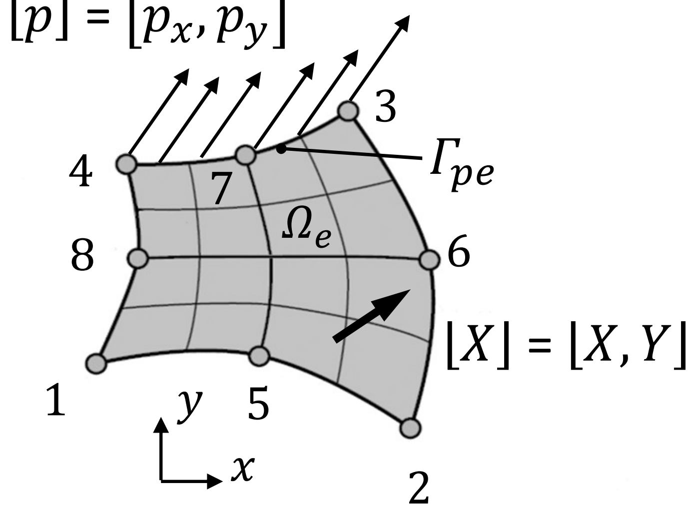
potential energy of loading in a finite element:

$$W_e = \int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 1}{\{u\}}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 1}{\{u\}}\, d\Gamma_{pe} =$$

$$\underset{2\times 1}{\{u\}} = \underset{2\times 16}{[N]} \underset{16\times 1}{\{q\}_e}$$

$$= \left(\int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 16}{[N]}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]}\, d\Gamma_{pe}\right) \underset{16\times 1}{\{q\}_e} =$$

$$= (\underset{1\times 16}{\lfloor F^X \rfloor_e} + \underset{1\times 16}{\lfloor F^p \rfloor_e}) \underset{16\times 1}{\{q\}_e} = \underset{1\times 16}{\lfloor F \rfloor_e} \underset{16\times 1}{\{q\}_e}$$

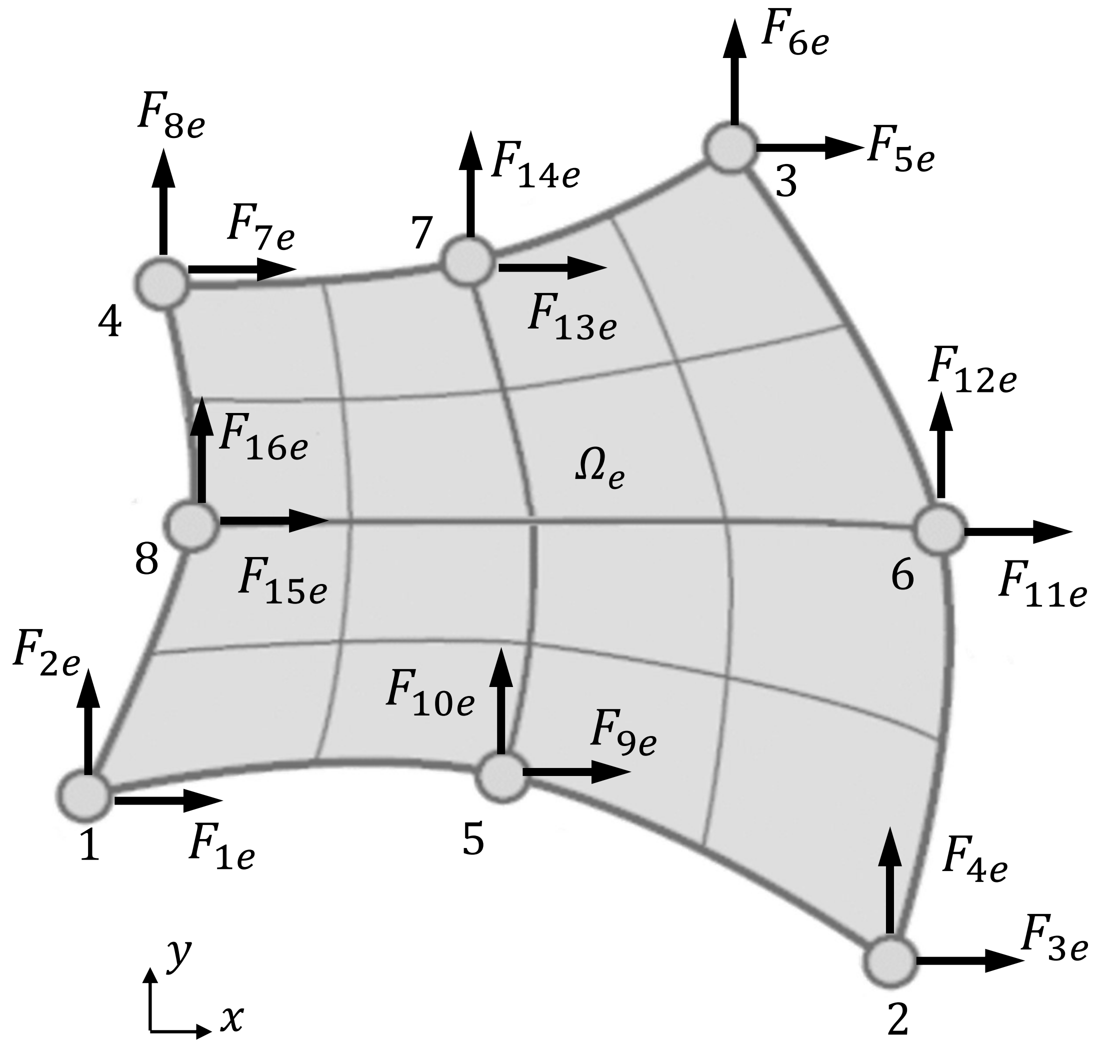


$$\underset{1\times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \underset{1\times 2}{\lfloor X(\xi,\eta) \rfloor} \underset{2\times 16}{[N(\xi,\eta)]}\, det[J]\, d\xi d\eta$$

$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} d\Gamma_{pe}$$

Equivalent load vector in the 8-node quadrilateral element

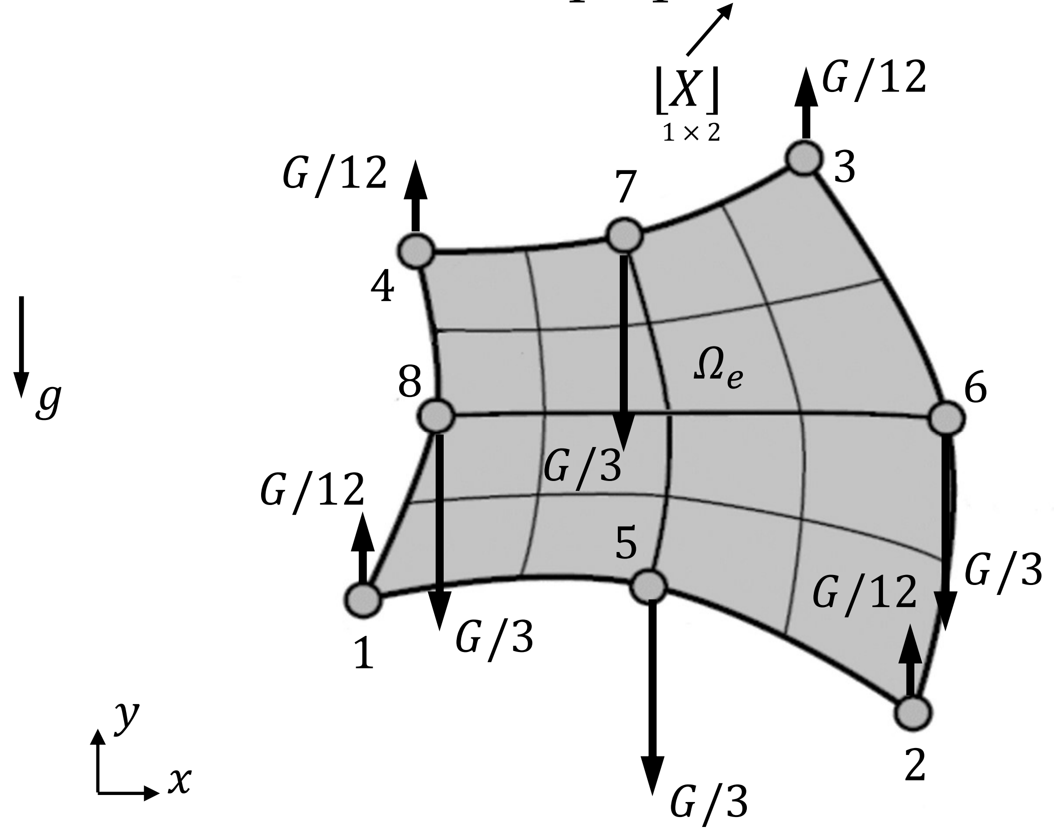
$\lfloor F \rfloor_e$
$_{16 \times 1}$



Example. Equivalent load vector due to mass forces (gravity load)

gravity load:

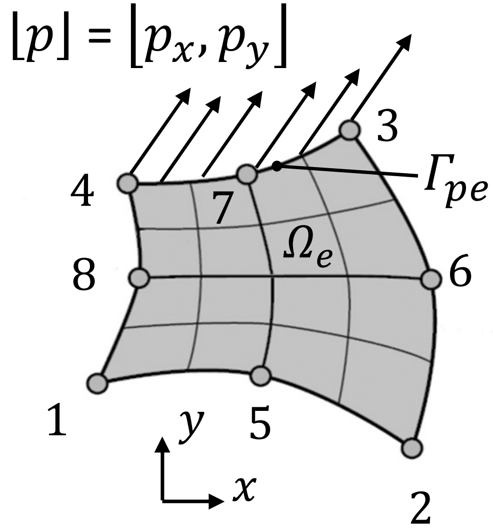
$$\underset{1\times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor 0, -\rho g \rfloor [\underset{2\times 16}{N(\xi,\eta)}] \, det[J] \, d\xi d\eta$$



$G = \rho g \Omega_e \ (N)$

$$\underset{1\times 16}{\lfloor F \rfloor_e} = \left\lfloor 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, \right\rfloor_e$$

Equivalent load vector due to surface load



equivalent load vector due to surface load:

$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} d\Gamma_{pe} = t_e \int_0^l \lfloor p \rfloor [N] ds$$

$$= t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} ds = t_e \int_{-1}^{1} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} \frac{ds}{d\xi} d\xi$$

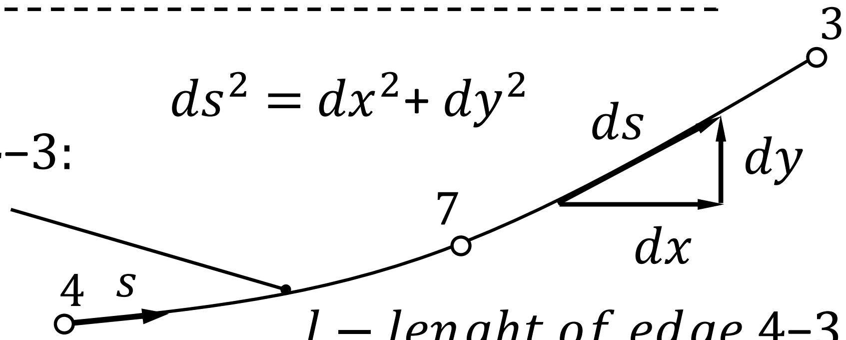
$$\frac{ds^2}{d\xi^2} = \frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} \quad \rightarrow \frac{ds}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = t_e \int_{-1}^{1} \lfloor p_x, p_y \rfloor \underset{2\times 16}{[N]} \sqrt{\left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{x_i\}_e}\right)^2 + \left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{y_i\}_e}\right)^2} d\xi$$

$(calculated\ numerically)$

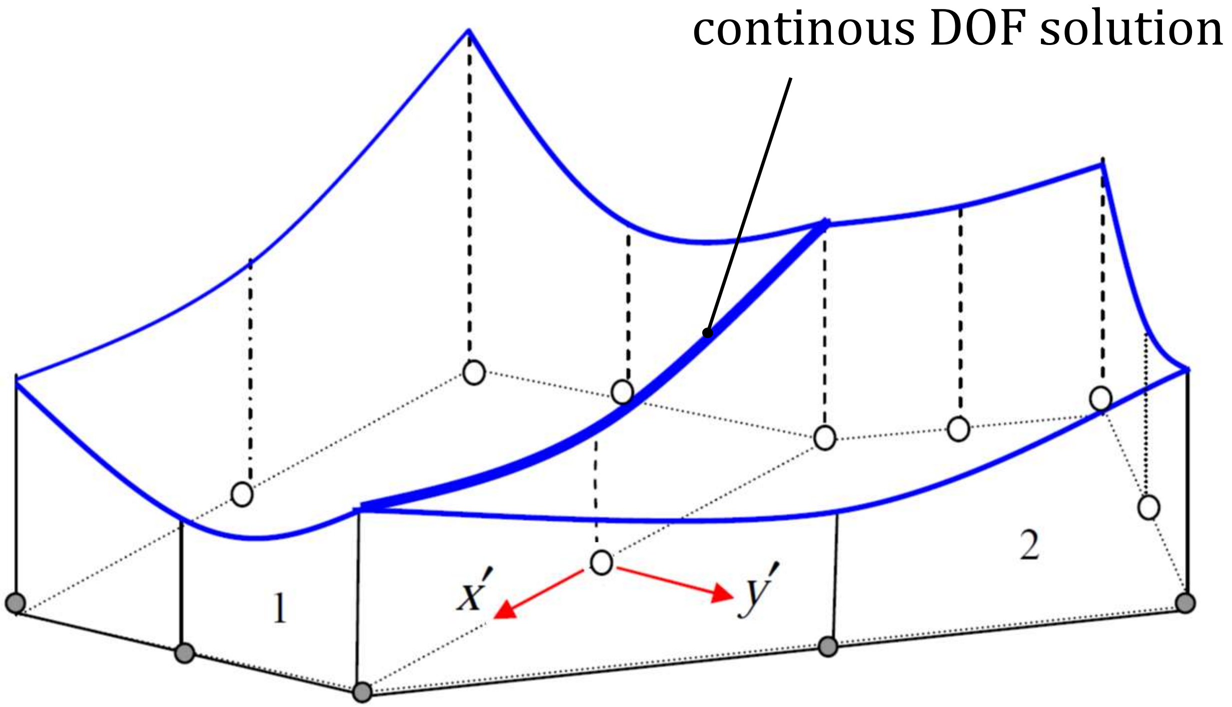
$$ds^2 = dx^2 + dy^2$$

on edge 4–3:
$\eta = 1$



$l - lenght\ of\ edge\ 4–3$

Results on the edge between two 8-node FEs



$$\left.\frac{\partial u}{\partial x'}\right|_1 = \left.\frac{\partial u}{\partial x'}\right|_2 \qquad \left.\frac{\partial u}{\partial y'}\right|_1 \neq \left.\frac{\partial u}{\partial y'}\right|_2 \Rightarrow \begin{matrix}(\varepsilon_{x'})_1 = (\varepsilon_{x'})_2 \\ (\varepsilon_{y'})_1 \neq (\varepsilon_{y'})_2\end{matrix} \Rightarrow (\sigma_{ij})_1 \neq (\sigma_{ij})_2$$

not continous element solution

Example. 2D model of a centilever beam (8-node FEs)